Fibonacci Exploration

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1 Introduction

The Fibonacci sequence, discovered by Leonardo da Pisa (or Fibonacci), is a recursively defined sequence in which the first two terms are both 1 and each successive term is the sum of the two previous terms. One of the most well-known sequences in mathematics, the Fibonacci sequence contains many patterns that can be found through careful observation and deep thinking. This sequence also has many applications, both within and outside of mathematics (particularly, in nature). For these reasons, there are countless possibilities for exploration in relation to this sequence. In our project, we examined several topics related to the Fibonacci numbers, namely arithmetic, modular cycles, and the relationships between this sequence and other special numbers.

2 Exploration Categories

2.1 Fibonacci Arithmetic

Proposition 1: *Consecutive Fibonacci Numbers are Coprime.*

Proof:

Base Case: $F_1 = 1, F_2 = 1$

$$(F_1, F_2) = 1$$

Assume: $(F_n, F_{n+1}) = 1$
Prove: $(F_{n+1}, F_{n+2}) = 1$
Since: $F_{n+2} = F_{n+1} + F_n$

$$(F_{n+1}, F_{n+2}) = (F_{n+1}, F_{n+1} + F_n)$$

$$(F_{n+1}, F_{n+1} + F_n) = (F_{n+1}, F_n) = 1 \ (By\ Induction\ Assumption)$$

$$Thus: (F_{n+1}, F_{n+1} + F_n) = 1$$

$$(F_{n+1}, F_{n+2}) = 1$$

Therefore, $for\ n > 0\ ,\ (F_n, F_{n+1}) = 1$ □

Proposition 2: $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}\ ,\ m > 1,\ n \geq 1,\ m, n\ \in \mathbb{N}.$

Proof: Keeping m *fixed, we apply induction on* n.

Let $n = 1$

$$\begin{aligned} \therefore F_{m+1} &= F_{m-1}F_1 + F_mF_2 \\ &= F_{m-1} \cdot 1 + F_m \cdot 1 \\ &= F_{m-1} + F_m \end{aligned}$$

$\Rightarrow$ The statement holds for $n = 1$

Let the statement hold for for n= $1, 2, \ldots, k$

$$\begin{aligned} \therefore F_{m+(k-1)} &= F_{m-1}F_{k-1} + F_mF_k \\ F_{m+k} &= F_{m-1}F_k + F_mF_{k+1} \end{aligned}$$

$$\begin{aligned} \therefore\ \ F_{m+k-1} + F_{m+k} &= F_{m-1}(F_{k-1} + F_k) + F_m(F_k + F_{k+1}) \\ F_{m+(k+1)} &= F_{m-1}F_{k+1} + F_mF_{(k+1)+1} \end{aligned}$$

$\therefore$ When the statement holds for $k \in \mathbb{N}$, it also holds for $(k+1) \in \mathbb{N}$. Also, the statement holds for 1.
$\therefore$ The statement holds $\forall n \in \mathbb{N}$.

$$\therefore F_{m+n} = F_{m-1}F_n + F_mF_{n+1}\ ,\ m > 1,\ n \geq 1,\ m, n\ \in \mathbb{N} \quad \square$$

Proposition 3: $F_m | F_{mn},\ m \geq 1,\ n \geq 1,\ m, n\ \in \mathbb{N}$.

Proof: Keeping m fixed, we apply induction on n.

Let n= 1

$$\therefore F_m | F_m \text{ , which is true}$$

$$\Rightarrow \text{The statement holds for } n = 1$$

Let the statement hold for for n = 1,2,...,k

$$\begin{aligned} &\therefore F_m | F_{mk} \\ &\quad F_{m(k+1)} = F_{mk+m} = F_{mk-1}F_m + F_{mk}F_{m+1} \\ &\quad F_m | F_m \text{ and } F_m | F_{mk} \\ &\Rightarrow F_m | (F_{mk} + F_{mk}F_{m+1}) \\ &\text{or, } F_m | F_{m(k+1)} \end{aligned}$$

$\Rightarrow$ The statement holds $\forall m, n \in \mathbb{N},\ m \geq 1,\ n \geq 1$.
$\therefore F_m | F_{mn},\ m \geq 1,\ n \geq 1,\ m, n\ \in \mathbb{N} \quad \square$

Proposition 4: If $m = nq + r$, $m, n \in \mathbb{N}$, $q, r \in \mathbb{N}$ s.t. $0 \leq r < n \Rightarrow (F_m, F_n) = (F_r, F_n)$

Proof:

$$F_m = F_{nq+r} \text{ (By Hypothesis)}$$
$$F_{nq+r} = F_{nq-1}F_r + F_{nq}F_{r+1}$$
$$\therefore (F_m, F_n) = (F_{nq+r}, F_n) = (F_{nq-1}F_r + F_{nq}F_{r+1}, F_n)$$
$$\text{Now, } F_n | F_{nq}$$
$$\therefore F_n | F_{nq}F_{r+1}$$
$$\text{We know, } (a+b, c) = (a, c), \text{ if } c|b$$
$$\therefore (F_{nq-1}F_r + F_{nq}F_{r+1}, F_n) = (F_{nq-1}F_r, F_n)$$
$$\text{Now, let } (F_{nq-1}, F_n) = \delta$$
$$\delta | F_n \;,\; F_n | F_{nq} \Rightarrow \delta | F_{nq}$$
$$\text{Also, } \delta | F_{nq-1}$$
$$\text{But } (F_{nq}, F_{nq-1}) = 1 \text{ (By Proposition (1))}$$
$$\Rightarrow \delta = 1$$
$$\therefore (F_{nq-1}, F_n) = 1$$
$$\text{If } (a, b) = 1, \text{ then } (ac, b) = (c, b)$$
$$\therefore (F_{nq-1}, F_n) = (F_r, F_n)$$
$$\Rightarrow (F_m, F_n) = (F_r, F_n), \text{ where } m = nq + r \;,\; m, n, q, r \in \mathbb{N} \text{ s.t. } 0 \le r < n \quad \square$$

Proposition 5: $(F_m, F_n) = F_{(m,n)},\ m \ge 1,\ n \ge 1.$

Proof:

$$\text{Let } m = nq_1 + r_1, \quad q_1, r_1 \in \mathbb{N}, \quad 0 \le r_1 < n$$
$$n = r_1 q_2 + r_2, \quad q_2, r_2 \in \mathbb{N}, \quad 0 \le r_2 < r_1$$
$$r_1 = r_2 q_3 + r_3, \quad q_3, r_3 \in \mathbb{N}, \quad 0 \le r_3 < r_2$$
$$.$$
$$.$$
$$.$$
$$r_{n-2} = r_{n-1}q_n + r_n, \quad q_n, r_r \in \mathbb{N}, \quad 0 \le r_n < r_{n-1}$$
$$r_{n-1} = r_n q_{n+1} + 0, \quad q_{n+1} \in \mathbb{N}, \quad \leftarrow \text{(i)}$$
$$\Rightarrow (F_m, F_n) = (F_{r_1}, F_n) = (F_{r_2}, F_{r_1}) = \cdots = (F_{r_{n-1}}, F_{r_n})$$
$$\text{However, } r_n | r_{n-1} \text{ (By (i))}$$
$$\Rightarrow F_{r_n} | F_{r_{n-1}}$$
$$\therefore (F_{r_n}, F_{r_{n-1}}) = F_{r_n}$$
$$\Rightarrow (F_m, F_n) = F_{r_n}$$

$$\text{But } (m, n) = r_n$$
$$\therefore (F_m, F_n) = F_{(m,n)} \quad \square$$

Proposition 6: ${F_n}^2 + {F_{n+1}}^2 = F_{2n+1}$

Proof:

Base Case: $n = 1$

$$\begin{aligned} {F_1}^2 + {F_2}^2 &= 1^2 + 1^2 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

$$\Rightarrow \text{The statement holds for } n = 1$$

Assume: ${F_n}^2 + {F_{n+1}}^2 = F_{2n+1}$
Prove: ${F_{n+1}}^2 + {F_{(n+1)+1}}^2 = F_{2(n+1)+1}$

${F_n}^2 + {F_{n+1}}^2 = F_{2n+1}$ (By Assumption)

$$\begin{aligned} {F_{n+1}}^2 &= F_{2n+1} - {F_n}^2 \\ &= (F_{2n+3} - F_{2n+2}) - {F_n}^2 \\ &= F_{2n+3} - F_{2n+2} - (F_{n+2} - F_{n+1})^2 \\ &= F_{2n+3} - F_{2n+2} - ({F_{n+2}}^2 - 2F_{n+2}F_{n+1} + {F_{n+1}}^2) \\ &= F_{2n+3} - F_{2n+2} - {F_{n+2}}^2 + 2F_{n+2}F_{n+1} - {F_{n+1}}^2 \\ &= F_{2n+3} - {F_{n+2}}^2 - F_{2n+2} + (F_{n+3} + F_n)F_{n+1} - {F_{n+1}}^2 \\ &= F_{2n+3} - {F_{n+2}}^2 - F_{2n+2} + F_{n+3}F_{n+1} \\ &\quad + F_nF_{n+1} - {F_{n+1}}^2 \\ &= F_{2n+3} - {F_{n+2}}^2 - F_{2n+2} + (F_{n+2} + F_{n+1})F_{n+1} + \\ &\quad F_nF_{n+1} - {F_{n+1}}^2 \\ &= F_{2n+3} - {F_{n+2}}^2 - F_{2n+2} + F_{n+2}F_{n+1} + {F_{n+1}}^2 \\ &\quad + F_nF_{n+1} - {F_{n+1}}^2 \\ &= F_{2n+3} - {F_{n+2}}^2 - F_{2n+2} + F_{n+2}F_{n+1} + F_nF_{n+1} \\ &= F_{2n+3} - {F_{n+2}}^2 - F_{(n+1)+(n+1)} + F_{n+2}F_{n+1} + F_nF_{n+1} \end{aligned}$$

We know, $F_{(n+1)+(n+1)} = F_nF_{n+1} + F_{n+1}F_{n+2}$ (By Proposition ②)

$$\begin{aligned} \therefore\ {F_{n+1}}^2 &= F_{2n+3} - {F_{n+2}}^2 - (F_nF_{n+1} + F_{n+1}F_{n+2}) + F_{n+2}F_{n+1} + F_nF_{n+1} \\ &= F_{2n+3} - {F_{n+2}}^2 - F_nF_{n+1} - F_{n+2}F_{n+1} + F_{n+2}F_{n+1} + F_nF_{n+1} \\ &= F_{2n+3} - {F_{n+2}}^2 \end{aligned}$$

$$\Rightarrow {F_{n+1}}^2 + {F_{n+2}}^2 = F_{2n+3}$$
$$\Rightarrow {F_{n+1}}^2 + {F_{(n+1)+1}}^2 = F_{2(n+1)+1} \quad \square$$

Proposition 7: ${F_{n+2}}^2 - {F_n}^2 = F_{2n+2}$

Proof:

$$\begin{aligned}\text{We know, } F_{2n+2} &= F_{(n+1)+(n+1)} = F_nF_{n+1} + F_{n+1}F_{n+2} \quad \text{(By Proposition ②)}\\ &= F_n(F_{n+2} - F_n) + (F_{n+2} - F_n)F_{n+2}\\ &= F_nF_{n+2} - {F_n}^2 + {F_{n+2}}^2 - F_nF_{n+2}\\ &= {F_{n+2}}^2 - {F_n}^2 \quad \square\end{aligned}$$

Proposition 8: $\sum_{i=1}^{n} F_{2i-1} = F_{2n}$

$Proof$:

$$F_{2k-1} : 1, 2, 3, 5, 13, 34, \dots$$

$$F_{2k} : 1, 3, 8, 21, 55 \dots$$

Let $n = 1$:

$$\therefore F_{2\cdot 1} = F_2 = 1$$

$$\sum_{i=1}^{1} F_{2i-1} = F_{2\cdot 1} - 1 = F_1 = 1$$

$$\therefore \sum_{i=1}^{n} F_{2i-1} = F_{2n} \text{ for } n = 1$$

Using induction let the formula hold for $n = 1$ to $n = k$, $k \in \mathbb{N}$.

$$\therefore F_{2k} = \sum_{i=1}^{k} F_{2i-1}$$

$$F_{2k} = F_{2k-1} + F_{2k-3} + \cdots + F_3 + F_1$$

$$\therefore F_{2k+1} + F_{2k} = F_{2k+1} + F_{2k-1} + F_{2k-3} + \cdots + F_3 + F_1$$

or,

$$F_{2k+2} = \sum_{i=1}^{k+1} F_{2i-1}$$

or,

$$F_{2(k+1)} = \sum_{i=1}^{k+1} F_{2i-1}$$

$\therefore$ The formula holds for $n = k + 1$ using our induction hypothesis so the formula holds $\forall n \in \mathbb{N}$ $\square$

Proposition 9: All natural numbers can be expressed as sum of distinct Fibonacci numbers.

Proof :
The sequence of Fibonacci numbers continues infinitely. So we can show that all the natural numbers from 1 to some F_{n-2} $(n > 2)$ and also 1 to F_{n-1} can be expressed as the sum of distinct Fibonacci numbers then we can show that all the natural numbers up to F_n can also be expressed as the sum of distinct Fibonacci numbers.

$$1 = F_1 = 1$$

$$2 = F_3 = 2$$

$$3 = F_4 = 3$$

$$4 = F_4 + 1 = 3 + 1$$

$$5 = F_5 = 5$$

For $n = 1$, F_1 can be expressed as the sum of distinct Fibonacci numbers namely 1. Let $\exists n = k$, s.t. all natural numbers from 1 to F_{k-1} can be expressed as the sum of distinct Fibonacci numbers. Now, $F_k = F_{k-1} + F_{k-2}$
$\therefore$ All the natural numbers from 1 to F_{k-2} can also be expressed as the sum of distinct Fibonacci numbers.
$\therefore$ All numbers from F_{k-1} to F_k can be expressed as $F_{k-1} + 1, F_{k-1} + 2, \ldots, F_{k-1} + F_{k-2}$.
Any m, s.t. $1 \leq m \geq F_{k-2}$ can be expressed as the sum of distinct Fibonacci numbers.
$\therefore$ Any $F_{k-1} + m$ can also be expressed in such a way.
$\therefore$ Such an expression is valid when $n = 1$ and it is valid for $n = k$, when it holds for $n = k$
$\rightarrow$ All natural numbers can be expressed as the sum of distinct Fibonacci Numbers. □

2.2 Cycles of Fibonacci modulo natural numbers

Definition 1: R_n: *The number of elements in one cycle of the sequence of remainder of the Fibonacci sequence modulo n*
Proposition 1: *The sequence of remainder of Fibonacci sequence modulo a natural numbers repeats for all natural numbers(except 1). The repetition always starts with 1,1.*

Proof: Consider the sequence of Fibonacci numbers modulo n, every element of this sequence is in $\mathbb{Z}_n$
Let $\{R_i\}$ be the sequence of remainders of the Fibonacci sequence modulo n, where R_i is the ith element of $\{R_i\}$

$$\{R_i\} = \{0, 1, 2, 3, \cdots, (n-1)\}$$

There are n elements in $\{R_i\}$ in total
Clearly, the first two elements of $\{R_i\}$ are 1 and 1 respectively.

Let

$$S_i := \{R_{2i-1}, R_{2i}\}$$

Since there are only n possible different R_i, there are only n^2 different S_i.
hence,by Pigeonhole Principle, S_{n^2+1} must be the same as one of S_1 to S_{n^2}

Let

$$S'_i := \{S_i, S_{i+1}\}$$

since there are only n^2 different S_i, there are only n^4 different S'_i
so,by Pigeonhole Principle, S'_{n^4+1} must be the same as one of S'_1 to S'_{n^4}

Let

$$S'_{n^4+1} = S'_j, j \in \mathbb{N}$$

So:

$$\{S_{n^4+1}, S_{n^4+2}\} = \{S_j, S_{j+1}\}$$

Since a fixed S_k ($k \in \mathbb{N}$)decides the rest of the sequence(two consecutive elements determine the rest of the sequence by the definition of the Fibonacci sequence) , once S_k starts to repeat, R_i repeats.
$\Rightarrow \{R_i\}$ does repeat
Since there are only a finite number of permutation, the permutation $\{1, 1\}$ must exist again
$\Rightarrow \{R_i\}$ repeats from the beginning two numbers: 1,1

Proposition 2: $\forall a, b \in \mathbb{N}, R_{[a,b]} = [R_a, R_b]$

Proof: Suppose

$$(a, b) = d, d \in \mathbb{N}$$

Then

$$a = d \cdot a', b = d \cdot b', a', b' \in \mathbb{N}, (a', b') = 1$$

Since

$$[a, b] \cdot (a, b) = a \cdot b$$

So

$$[a, b] \cdot d = a \cdot b$$

Hence

$$[a, b] = ab/d$$

Since R_a, R_b are, respectively, the length of the cycle of F_n modulo a and F_n modulo b
so $\forall k \in \mathbb{N}$

$$F_{k \cdot R_a} \equiv 0 \ (mod\ a) \leftarrow \textcircled{1}$$

$$F_{k \cdot R_b} \equiv 0 \ (mod\ b) \leftarrow \textcircled{1}$$

According to definition, we have:

$$F_{k \cdot R_{ab/d}} \equiv 0 \ (mod\ ab/d)$$

$$\frac{ab}{d} = a \cdot \frac{b}{d} = a \cdot b'$$
$$F_{k \cdot R_{ab/d}} \equiv 0 \ (mod\ a \cdot b')$$
$$a \cdot b' | F_{k \cdot R_{ab/d}}$$
$$a | F_{k \cdot R_{ab/d}}$$

By definition:

$$F_{k \cdot R_{ab/d}} \equiv 0 \ (mod\ a)$$

Similarly,

$$F_{k \cdot R_{ab/d}} \equiv 0 \ (mod\ b)$$

By ①

$$R_{ab/d} = k_1 \cdot R_a \ \ k_1 \in \mathbb{N}$$
$$R_{ab/d} = k_2 \cdot R_b \ \ k_2 \in \mathbb{N}$$
$$R_{ab/d} \text{ is a common multiple of } R_a,\ R_b$$

By definition of LCM

$$[R_a, R_b] \,| R_{ab/d}$$
$$[R_a, R_b] \in \mathbb{N}\ ,\ R_{ab/d} \in \mathbb{N}$$

So

$$[R_a, R_b] \le R_{ab/d}$$

Also, Since $[R_a, R_b]$ is a multiple of $R_a,\ R_b$, by ①

$$F_{[R_a,R_b]} \equiv 0 \ \ (mod\ a)$$
$$F_{[R_a,R_b]} \equiv 0 \ \ (mod\ b)$$
$$F_{[R_a,R_b]} \equiv 0 \ \ (mod\ \frac{ab}{d}) \leftarrow (Lemma\ 1)$$

By definition:

$$R_{ab/d} | \,[R_a, R_b]$$
$$R_{ab/d} \in \mathbb{N}\ ,\ [R_a, R_b] \in \mathbb{N}$$
$$R_{ab/d} \le [R_a, R_b]$$

hence

$$R_{ab/d} \le [R_a, R_b]\ ,\ \ R_{ab/d} \ge [R_a, R_b]$$
$$R_{ab/d} = [R_a, R_b]$$
$$[R_a, R_b] = R_{[a,b]}$$
$$QED$$

Proposition

$$F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

where $F_0 = 0$ and $F_1 = 1$

Proof. Recall the recursion used for the Fibonacci numbers:

$$F_n = F_{n-1} + F_{n-2}$$

We can take advantage of a property of recursions and their closed forms, namely, generating functions

The function for the Fibonacci recursion is therefore

$$F_n = F_{n-1} + F_{n-2}$$

$$x^2 = x + 1$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Therefore the closed form for the Fibonacci recursion is of the form

$$F_n = A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n$$

Since $F_0 = 0$

$$0 = A + B$$

$$A = -B$$

Since $F_1 = 1$

$$1 = A(\frac{1+\sqrt{5}}{2}) + B(\frac{1-\sqrt{5}}{2})$$

$$1 = (-B)(\frac{1+\sqrt{5}}{2}) + B(\frac{1-\sqrt{5}}{2})$$

$$B = -\frac{1}{\sqrt{5}}$$

$$A = \frac{1}{\sqrt{5}}$$

$$F_n = (\frac{1}{\sqrt{5}})(\frac{1+\sqrt{5}}{2})^n + (-\frac{1}{\sqrt{5}})(\frac{1-\sqrt{5}}{2})^n$$

$$F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

□

Definition L_n refers to the Lucas number sequence, a sequence recursively defined as

$$L_n = L_{n-1} + L_{n-2}$$

with $L_0 = 2$ and $L_1 = 1$

Proposition

$$L_n = (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$$

where $L_0 = 2$ and $L_1 = 1$

Proof. Consider the recursion used for the Lucas Numbers

$$L_n = L_{n-1} + L_{n-2}$$

This gives us the accompanying function

$$x^2 = x + 1$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Therefore the closed form for the Lucas recursion is of the form

$$L_n = A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n$$

Recall $L_0 = 2$

$$2 = A + B$$

Recall $L_1 = 1$

$$1 = A(\frac{1+\sqrt{5}}{2}) + B(\frac{1-\sqrt{5}}{2})$$

$$A = 1$$

$$B = 1$$

$$L_n = (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$$

□

Proposition

$$F_n^2 - F_{n+k}F_{n-k} = (-1)^{k+n}F_k^2$$

where $F_1 = 1$ and $F_2 = 1$ and $F_n = F_{n-1} + Fn - 2$ with $k \in \mathbb{N}$ such that $k < n$

Proof. Recall the closed form for the Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

where $F_1 = 1$ and $F_2 = 1$ and $F_n = F_{n-1} + Fn - 2$

Consider:

$$F_n^2 - F_{n+k}F_{n-k}$$

in closed form

$$F_n^2 - F_{n+k}F_{n-k} = (\frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n))^2 - (\frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{n+k} - (\frac{1-\sqrt{5}}{2})^{n+k}))$$

$$(\frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{n-k} - (\frac{1-\sqrt{5}}{2})^{n-k}))$$

Suppose

$$x = \frac{1+\sqrt{5}}{2}$$
$$y = \frac{1-\sqrt{5}}{2}$$

Thus,

$$F_n^2 - F_{n+k}F_{n-k} = (\frac{1}{\sqrt{5}}(x^n - y^n))^2 - (\frac{1}{\sqrt{5}}(x^{n+k} - y^{n+k}))(\frac{1}{\sqrt{5}}(x^{n-k} - y^{n-k}))$$

Now, consider:

$$(-1)^{k+n}F_k^2$$

in closed form

$$(-1)^{k+n}F_k^2 = (-1)^{k+n}(\frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^k - (\frac{1-\sqrt{5}}{2})^k))^2 = (-1)^{k+n}(\frac{1}{\sqrt{5}}(x^k - y^k))^2$$

Therefore, it is sufficient to prove that

$$(\frac{1}{\sqrt{5}}(x^n - y^n))^2 - (\frac{1}{\sqrt{5}}(x^{n+k} - y^{n+k}))(\frac{1}{\sqrt{5}}(x^{n-k} - y^{n-k})) = (-1)^{k+n}(\frac{1}{\sqrt{5}}(x^k - y^k))^2$$

$$(\frac{1}{\sqrt{5}})^2((x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k})) = (\frac{1}{\sqrt{5}})^2(-1)^{k+n}(x^k - y^k)^2$$
$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = (-1)^{k+n}(x^k - y^k)^2$$

Now, consider the left side:

$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k})$$

$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = x^{2n} + y^{2n} - 2(xy)^n - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k})$$
$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = x^{2n} + y^{2n} - 2(xy)^n - (x^{2n} - x^{n+k}y^{n-k} - x^{n-k}y^{n+k} + y^{2n})$$
$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = x^{2n} + y^{2n} - 2(xy)^n - x^{2n} + x^{n+k}y^{n-k} + x^{n-k}y^{n+k} - y^{2n}$$
$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = -2(xy)^n + x^{n+k}y^{n-k} + x^{n-k}y^{n+k}$$
$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = -2(xy)^n + (xy)^{n-k}(x^{2k} + y^{2k})$$

Recall

$$x = \frac{1+\sqrt{5}}{2}$$
$$y = \frac{1-\sqrt{5}}{2}$$

Thus,

$$xy = (\frac{1+\sqrt{5}}{2})(\frac{1-\sqrt{5}}{2}) = -1$$

Therefore,

$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = -2(-1)^n + (-1)^{n-k}(x^{2k} + y^{2k})$$

$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = -2(-1)^n 1^k + (-1)^{n-k} 1^k (x^{2k} + y^{2k})$$

$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = -2(-1)^n(-1)^{2k} + (-1)^{n-k}(-1)^{2k}(x^{2k} + y^{2k})$$

$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = -2(-1)^{2k+n} + (-1)^{n+k}(x^{2k} + y^{2k})$$

$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = -2(-1)^{2k+n} + (-1)^{n+k}x^{2k} + (-1)^{n+k}y^{2k}$$

$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = (-1)^{k+n}(x^{2k} + y^{2k} - 2(-1)^k)$$

$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = (-1)^{k+n}(x^{2k} + y^{2k} - 2(xy)^k)$$

$$(x^n - y^n)^2 - (x^{n+k} - y^{n+k})(x^{n-k} - y^{n-k}) = (-1)^{k+n}(x^k - y^k)^2$$

Thus,

$$F_n^2 - F_{n+k}F_{n-k} = (-1)^{k+n}F_k^2$$

□

Proposition $\varphi^k = F_{k-1}\varphi + F_{k-2}$ where $F_0 = 1$, $F_1 = 1$, $k \geq 2$

Proof. We can prove this statement using induction

Base Case:

$$k = 2$$

$$\varphi^2 = F_1\varphi + F_0$$

$$\varphi^2 = \varphi + 1$$

This is true, as it is a property of φ

Inductive Step:

Assume $\varphi^k = F_{k-1}\varphi + F_{k-2}$ is true

Prove $\varphi^{k+1} = F_k\varphi + F_{k-1}$

Consider

$$\varphi^{k+1}$$

$$\varphi^{k+1} = \varphi^k * \varphi$$

$$\varphi^{k+1} = (F_{k-1}\varphi + F_{k-2}) * \varphi$$

$$\varphi^{k+1} = F_{k-1}\varphi^2 + F_{k-2}\varphi$$

Since $\varphi^2 = \varphi + 1$,

$$\varphi^{k+1} = F_{k-1}\varphi^2 + F_{k-2}\varphi = F_{k-1} + (F_{k-1} + F_{k-2})\varphi$$

$$\varphi^{k+1} = F_{k-1}\varphi^2 + F_{k-2}\varphi = F_{k-1} + F_k\varphi$$

$$\varphi^{k+1} = F_k\varphi + F_{k-1}$$

□

Proposition

$\forall x \in \mathbb{N}$, $n \geq 1$, $\exists$ 1 or 2 F_n in the interval $[x, 2x]$ with $F_0 = 1$ and $F_1 = 1$

Proof. Notice that any natural number x is between two consecutive Fibonacci numbers F_n and F_{n+1}

$$F_n \leq x \leq F_{n+1}$$

$$0 \leq x - F_n \leq F_{n+1} - F_n$$

Note that since $n \geq 1$, $F_n, F_{n+1} \neq 1$. Thus, $F_n \neq F_{n+1}$, and $F_n \leq x \leq F_{n+1}$ is actually $F_n < x \leq F_{n+1}$ or $F_n \leq x < F_{n+1}$

Part 1: Prove Existence

First, assume that there exists no Fibonacci number in the interval $[x, 2x]$. Thus, since $2x$ must now be in the interval (x, F_{n+1}),

$$2x - x < F_{n+1} - x$$

$$2x - x < F_n + F_{n-1} - x$$

$$x - F_n < F_{n-1} - x$$

Recall that $x - F_n \geq 0$. In other words, the left hand side is a non-negative integer However, since

$$x \geq F_n \geq F_{n-1}$$

$$x \geq F_{n-1}$$

$$F_{n-1} - x \leq 0$$

Thus, since the right hand side is a non-negative integer, and the left hand side is a non-positive integer,

$$x - F_n = F_{n-1} - x = 0$$

Thus,

$$x = F_n = F_{n-1}$$

This is a contradiction to our assumption that there does not exist Fibonacci numbers in the interval $[x, 2x]$. Thus, there must always exist at least one Fibonacci number in that interval

Part 2: Prove a maximum of two Fibonacci numbers in that interval

Consider the inequality

$$F_n \leq x \leq F_{n+1}$$

Since we proved existence of at least one Fibonacci number in the interval,

$$F_n \leq x \leq F_{n+1} \leq 2x$$

Consider

$$F_n \leq x$$

$$F_n + F_{n+1} \leq x + F_{n+1}$$

$$F_{n+2} \leq x + F_{n+1}$$

Consider $x = F_{n+1}$
$$F_{n+2} \leq x + x$$
$$F_{n+2} \leq 2x$$
(Note that since $x = F_{n+1}$, the equation $F_{n+1} \leq F_{n+2} \leq 2x \Rightarrow x \leq F_{n+2} \leq 2x$) Since $x = F_{n+1}$ is completely legitimate by our definitions, we have proven that F_{n+2} is also in our interval, and thus, it is possible to have two Fibonacci numbers in the interval $[x, 2x]$

Now, assume there exists an x such that there are three Fibonacci numbers in the interval $[x, 2x]$. Thus,
$$F_n \leq x \leq F_{n+1} \leq F_{n+2} \leq F_{n+3} \leq 2x$$
Consider $F_{n+3} \leq 2x$
$$F_{n+1} + F_{n+2} \leq 2x$$
$$2F_{n+1} + F_n \leq 2x$$
Since $F_n \leq x$, and we are in natural numbers,
$$2F_{n+1} + F_n \leq 2F_n$$
$$2F_{n+1} \leq F_n$$
This is a contradiction, as
$$F_{n+1} \geq F_n$$
Thus, our assumption was incorrect that there exists three Fibonacci numbers in the interval $[x, 2x]$. Therefore, there exists one or two Fibonacci numbers within the interval $[x, 2x]$ with x in the natural numbers and $n \geq 1$ □

2.3 Lemma

$\forall k \in \mathbb{Z}, a \in \mathbb{N}, b \in \mathbb{N}$,
If
$$k \equiv 0 \pmod a, k \equiv 0 \pmod b$$
then
$$k \equiv 0 \pmod{[a, b]}$$

Proof:
Suppose
$$k \equiv 0 \pmod a, k \equiv 0 \pmod b$$
By definition of moduolo
$$a|k, b|k$$
Let
$$k = a \cdot k_1, k = b \cdot k_2, k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z}$$
Suppose
$$(a, b) = d, d \in \mathbb{N}$$

Then

$$a = d \cdot a', b = d \cdot b', a' \in \mathbb{N}, b' \in \mathbb{N}, (a', b') = 1$$

Since

$$[a, b] \cdot (a, b) = a \cdot b = (a' \cdot d) \cdot (b' \cdot d)$$

So

$$[a, b] \cdot d = a'b'd^2$$

Hence

$$[a, b] = a'b'd$$

$$\Rightarrow k = a'dk_1, k = b'dk_2$$

Hence

$$a'k_1 = b'k_2$$

According to the Fundamental Theorem of Arithmetic
Since

$$a'k_1 | b'k_2, (a', b') = 1$$

Then

$$a' | k_2$$

Let

$$k_2 = k' \cdot a', k' \in \mathbb{Z}$$

Then

$$k = b \cdot k_2 = (b'd) \cdot (a'k') = a'b'dk'$$

Hence

$$a'b'd | k$$

$$\Rightarrow [a, b] | k$$

□

2.4 Other Conjectures

1. If p is a rational prime, $m \in \mathbb{N}$ then $R_{p^m} = R_p * p^{m-1}$
2. If p is a rational prime, and p $\equiv 3$ or $7 \ (mod\ 10)$, then $R_p = \frac{2(p+1)}{n}$ for some $n \in \mathbb{N}$
3. If p is a rational prime, and p $\equiv 1$ or $9 \ (mod\ 10)$, then $R_p = \frac{(p-1)}{n}$ for some $n \in \mathbb{N}$
4. $\forall n \in \mathbb{Z}^{\geq 0}$, $t \in \mathbb{N}$ and $t \geq 3$ then: $2^t | F_{3*2^{t-2}(2n+1)}$ and $2^{t+1} \nmid F_{3*2^{t-2}(2n+1)}$
5. $\forall n \in \mathbb{N}$ If $n \nmid 2$ and $n > 3$ then $\exists p, k \in \mathbb{N}$ where p is prime and $p^k | F_n$ s.t. $4n = R_{p^k}$
6. $\forall n \in \mathbb{N}$ If $n | 2$ and $n > 2$ then $\exists p, k \in \mathbb{N}$ where p is prime and $p^k | F_n$ s.t. $2n = R_{p^k}$

7. If $F_1 = 0$ and $F_2 = 1$, (a_n, b_n, c_n) is a Pythagorean triple following:

$$(a_n, b_n, c_n) = (a_{n-1} + b_{n-1} + c_{n-1}, F_{2n-1} - b_{n-1}, F_{2n})$$

$$(a_3, b_3, c_3) = (4, 3, 5)$$
$$n \geq 4$$

8. If $F_0 = 0$ and $F_1 = 1$, (a_n, b_n, c_n) is a pythagorean triple following:

$$(a_n, b_n, c_n) = (2F_nF_{n-1}, F_{2n-1}, F_n^2 - F_{n-1}^2)$$

$$n \geq 3$$

9. $\forall n$, $\exists X, Y$ st $|X| = F_x$ and $|Y| = F_{x\pm1}$

$$XF_n + YF_{n+1} = 1$$

10. If $F_0 = 0$ and $F_1 = 1$ with $L_0 = 2$ and $L_1 = 1$
$\forall n$ and an arbitrary k ($k \leq n$)
If k is odd,

$$5F_kF_n = L_{n-k} + L_{n+k}$$
$$F_kL_n = F_{n-k} + F_{n+k}$$

If k is even,

$$L_nL_k = L_{n-k} + L_{n+k}$$
$$L_kF_n = F_{n-k} + F_{n+k}$$

11. $\forall n \in \mathbb{N}$, $\exists$ F_r and L_m st if F_r had r digits, and n has n' digits,

$$F_r = n * 10^{r-n'} + k$$

$$k \in \mathbb{N}$$
$$k < 10^{r-n'}$$

Likewise, if L_m has m digits,

$$L_m = n * 10^{m-n'} + k'$$

$$k' \in \mathbb{N}$$
$$k < 10^{m-n'}$$

12.

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$$

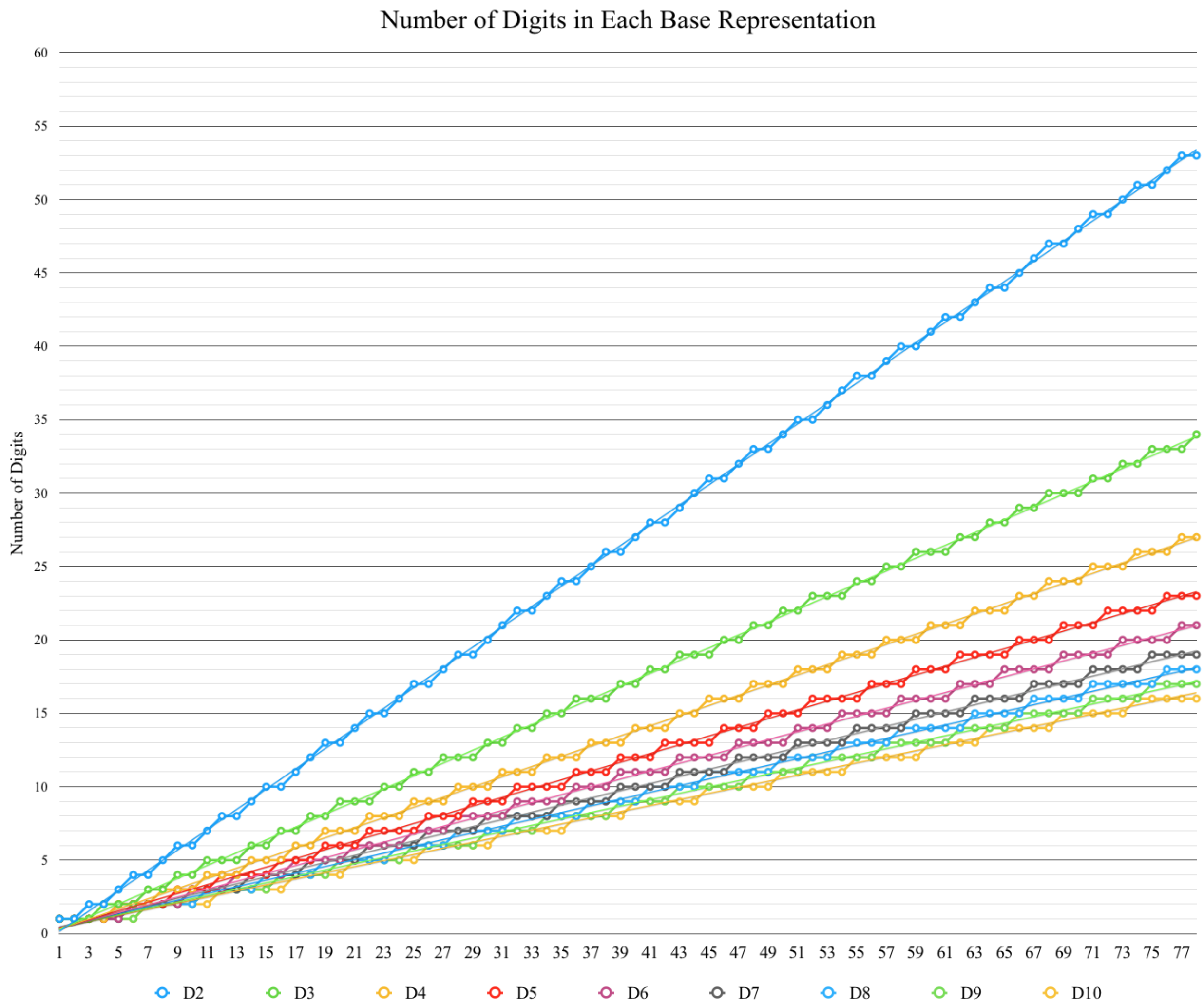

Figure 1: A graph of the increasing number of digits in different bases.

2.5 Base Representation Trend

The rates at which the number of digits of various base representations of Fibonacci numbers increase can be visualized through the following figure. After graphing the lines of best fit for the trend within each base, one can observe that the slopes of these lines seem to follow a negative exponential trend from base 2 to base 10.
Approximate equations of lines of best fit:

$$\begin{aligned}
\text{Base 2: } y &= 0.6915x - 0.5331 \\
\text{Base 3: } y &= 0.4363x - 0.1568 \\
\text{Base 4: } y &= 0.3466x - 0.0513 \\
\text{Base 5: } y &= 0.2978x + 0.0573 \\
\text{Base 6: } y &= 0.2679x + 0.0833 \\
\text{Base 7: } y &= 0.2441x + 0.2151 \\
\text{Base 8: } y &= 0.2296x + 0.2111 \\
\text{Base 9: } y &= 0.2177x + 0.1955 \\
\text{Base 10: } y &= 0.2078x + 0.2005
\end{aligned}$$